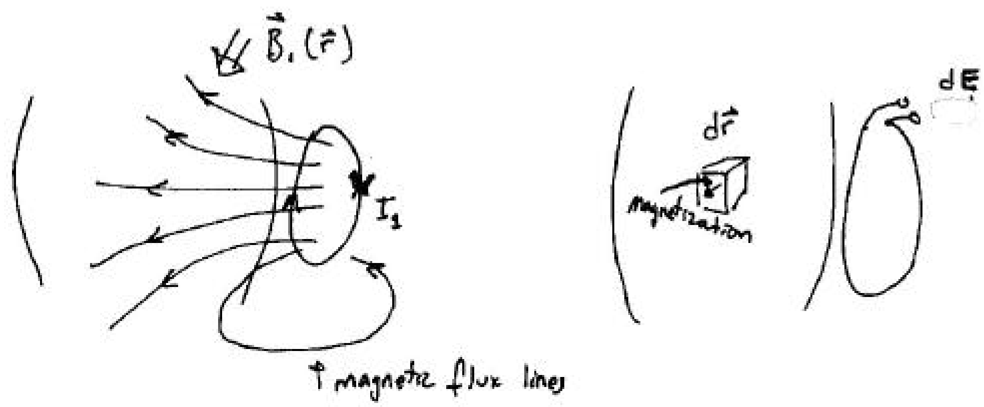
# Notes on MRI, Part II

## Signal Reception in MRI

The signal that we detect in MRI is a voltage induced in an RF coil by changes in magnetic flux from the precessing magnetization in the object. One expression for the voltage induced in a coil is:

$$E = -\frac{d\Phi}{dt}$$

where $\Phi$ is the flux in the coil. A common configuration is to use the same RF coil to transmit $B_1$ fields to the object and to receive signal from the magnetization. Assume, for a given coil configuration and current $I_1$, the RF field generated is $\mathbf{B_1}$. By the principle of reciprocity, the coil's receive sensitivity can be defined as $\mathbf{C_1} = \mathbf{B_1}/I_1$.




The incremental voltage produced by magnetization in an element $d\mathbf{r}$ is:

$$dE = -\left[\mathbf{C_1}(\mathbf{r}) \cdot \frac{\partial}{\partial t}\mathbf{M}(t)\right] d\mathbf{r}$$

Now suppose our magnetization comes from a precessing spins having magnetization $m_0$ in the presence of a magnetic field of size $B_0 + \Delta B$ (it has a resonant frequency of $w_0 + \Delta w$). That is:

$$\mathbf{M}(t) = \begin{bmatrix} m_0 \cos(w_0 + \Delta w)t \\ -m_0 \sin(w_0 + \Delta w)t \\ 0 \end{bmatrix}$$

and suppose the coil is position in the x-z plane making the sensitivity lines (flux lines) point in the y-direction:

$$\mathbf{C}_1 = \begin{bmatrix} 0 \\ C \\ 0 \end{bmatrix}$$

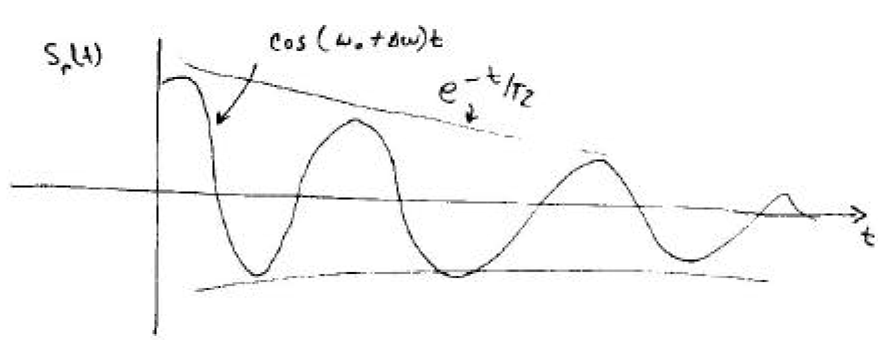
The voltage induced in the coil, which will become our received signal $s_r(t)$, can then be shown to be:

$$s_r(t) = dE = m_0(w_0 + \Delta w)C\cos(w_0 + \Delta w)t$$

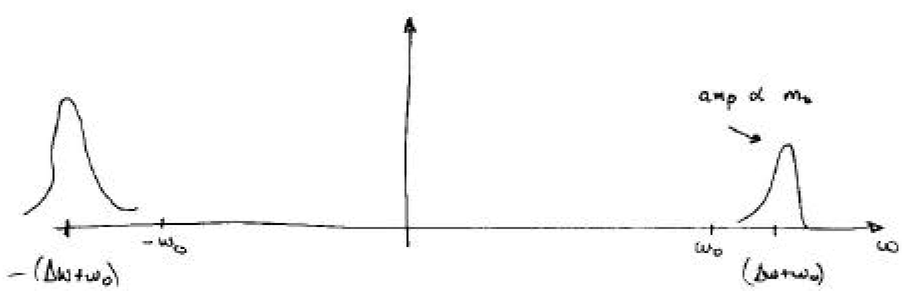
That is, the voltage on the wire will be a cosinusoidal variation at the resonant frequency and with an amplitude proportional to the coil sensitivity, the resonant frequency, and the size of the magnetization. We commonly make an assumption that $\Delta w$ is small relative to $w_0$ (a good assumption) and thus, the $(w_0 + \Delta w)$ term in the amplitude scaling is approximately a constant and can be absorbed into C:

$$m_0 C\cos(w_0 + \Delta w)t$$

If we include T2 relaxation, the received signal will look something like this:




This signal is known as the ***Free Induction Decay*** or FID (free – meaning it is not being driven by an RF pulse, induction – the action of a magnetic moment precessing around a magnetic field was first called (by Bloch) nuclear induction, and decay – meaning $T_2$ decay). If we take the Fourier transform of this received signal we will get approximately the following spectrum:



where the size of the spectral peaks at +/- $w_0 + \Delta w$ is proportional to $m_0$.

Now, suppose we groups of spin at different frequencies and amplitudes:

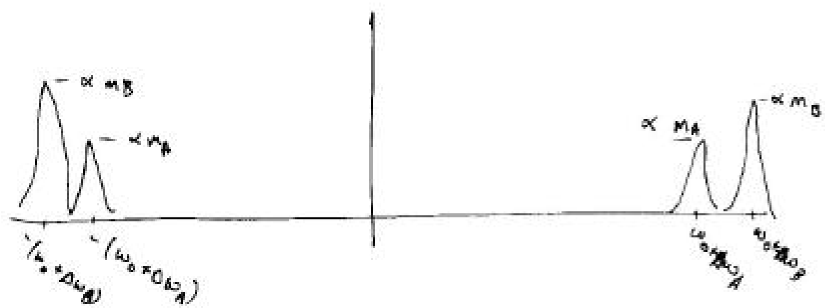
A: $m_A$, $\Delta w_A$

B: $m_B$, $\Delta w_B$

Now the voltage induced in the coil will be the sum of these two groups of spins:

$$s_r(t) = dE = m_A C\cos(w_0 + \Delta w_A)t + m_B C\cos(w_0 + \Delta w_B)t$$

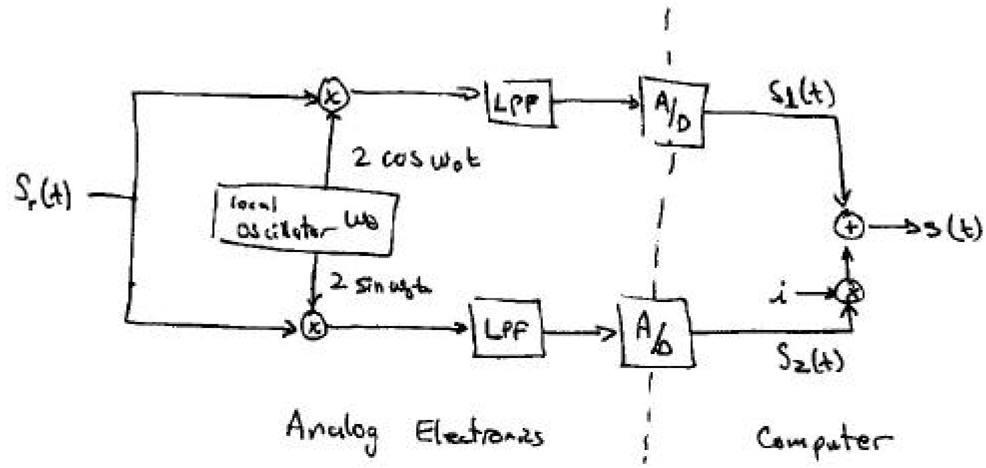
and the spectrum will have peaks at frequencies +/-($w_0 + \Delta w_{A,B}$) and amplitudes proportional to $m_{A,B}$.



In general, the voltage induced in the coil will be the summation (or integral) over the magnetization components that comprise the object we are imaging.

**Complex Demodulation**

The received signal, $s_r(t)$, is a real-valued voltage.  We transform this to a baseband signal using a complex demodulator, as shown here:




The components of this system are a local oscillator that supplies a cosine and a sine wave at frequency $\omega_0$. The received signal, $s_r(t)$, is multiplied by these signals and then low pass filtered (LPF) in order to produce a two signals that have a reduced bandwidth. These signals are then sampled using analog to digital (A/D) converters and then the sampled signals are combined on the computer to create a complex signal, $s(t)$.

We first look at the upper and lower channels of the complex demodulator for a single component of the received signal at location **r**. The upper channel of the demodulator yields:

$$s_1(t) = LPF\{2m_0 \cos(w_0 t + \Delta wt)\cos(w_0 t)\}$$

$$= LPF\{m_0[\cos(\Delta wt) + \cos(2w_0 t + \Delta wt)]\}$$

$$= m_0 \cos(\Delta wt)$$

and the lower channel yields:

$$s_2(t) = LPF\{2m_0 \cos(w_0 t + \Delta wt)\sin(w_0 t)\}$$

$$= LPF\{m_0[-\sin(\Delta wt) + \sin(2w_0 t + \Delta wt)]\}$$

$$= -m_0 \sin(\Delta wt)$$

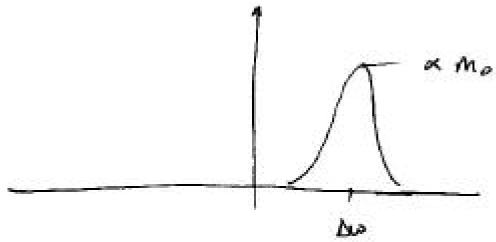
We can then construct the combined signal, $s(t)$:

$$s(t) = s_1(t) + is_2(t)$$

$$= m_0 \exp(-i\Delta wt)$$

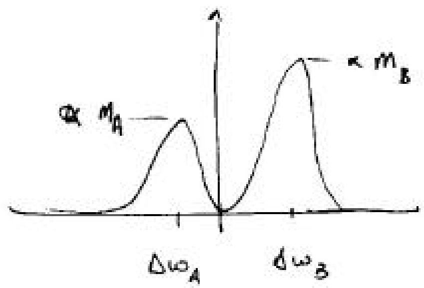
$$= m_0 \exp(-i\Delta wt)$$

This is a rotation in the complex plane at frequency $\Delta w$ and the resultant spectrum will look like:




Using a similar arguments, we can determine the baseband signal for the case of two objects (A and B, described above) as (we've let $C = 1$):

$$s(t) = m_A \exp(-i(w_0 + \Delta w_A)t) + m_B \exp(-i(w_0 + \Delta w_B)t)$$




*Important points!*

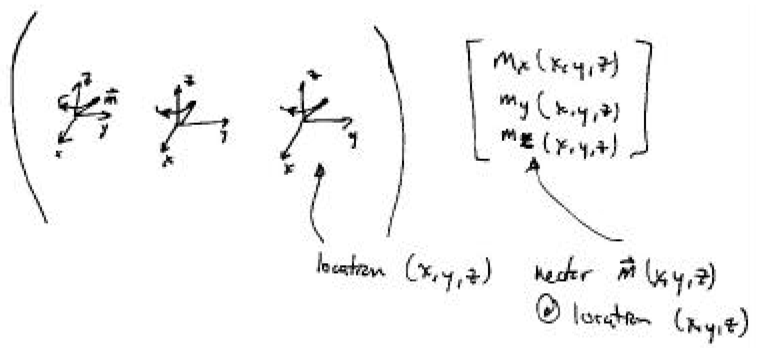
1. These complex signal are equivalent to the solutions to the Bloch equations in the rotating frame of reference. Through complex demodulation, we have access to the signal in the rotating frame where the frame frequency is determined by the local oscillator of the demodulator.
2. Since the signal $s(t)$ exists only on the computer, it is possible to have a complex signal.
3. The RF coil sums or integrates this signal from the entire object (or for the part of the object to which the coil is sensitive.

**Spatial and Temporal Variations**

We will now generalize our solution to the Bloch equations to functions in the object domain, for example:

$$m_{xy}(x, y, z, t) = m_x(x, y, z, t) + im_y(x, y, z, t)$$

Please note the distinction between the subscript *x*, which denotes the direction of a magnetization vector, and the argument *x*, which denotes the spatial location of that magnetization vector.



We also will allow the applied magnetic field to be a function of both space and time, but as before, we will first consider the case where the applied field is only in the *z* direction:

$$\mathbf{B}(x, y, z, t) = \left(B_0 + \Delta B(x, y, z, t)\right)\mathbf{k}$$

In imaging, we are typically dealing with just two of these spatial dimensions. It can be any of these two, but by convention we will use x and y. Thus, we will typically just use:

$$\mathbf{B}(x, y, t) = \left(B_0 + \Delta B(x, y, t)\right)\mathbf{k}$$

If **B** is constant, the solutions to the Bloch equations will then be:

$$m_{xy}(x, y, t) = m_0(x, y)\exp(-i(w_0 t + \Delta w(x, y)t))$$

where $\Delta w(x, y) = g\Delta B(x, y)$. In the rotating frame, the solution is:

$$m_{xy,rot}(x, y, t) = m_0(x, y)\exp(-i\Delta w(x, y)t)$$

In $m_{xy,rot}(x,y,t)$, the *x*,*y* in the argument refers to physical (*x*,*y*) locations in space, whereas the *xy* in the subscript refers to a mini-coordinate frame to describe direction of the magnetization vector at each point in space.

For a time-varying B field, the solution will take on a form similar to what we have seen before (in first set of notes on NMR):

$$m_{xy}(x, y, t) = m_0(x, y)\exp(-iw_0 t)\exp\left(-ig\int_0^t \Delta B(x, y, t')dt'\right)$$

and again, in the rotating frame, the solution is:

$$m_{xy,rot}(x, y, t) = m_0(x, y)\exp\left(-ig\int_0^t \Delta B(x, y, t')dt'\right)$$

**The Signal Equation.** Above, we described the voltage induced in a coil and further constructed a baseband signal that gave a representation of the signal in rotating frame. As discussed, the signal will be the sum or integral of all the spins the comprise the object. For a multidimensional object, the signal equation is the integral over the magnetization in the rotating frame (again, we will let the coil sensitivity, $C$ = 1):

$$s(t) = \iint m_{xy,rot}(x,y,t)dxdy$$

$$= \iint m_0(x,y)\exp\left(-i\Delta w(x,y)t\right)dxdy, \quad \text{or for a time-varying field :}$$

$$= \iint m_0(x,y)\exp\left(-ig\int_0^t \Delta B(x,y,t')dt'\right)dxdy$$

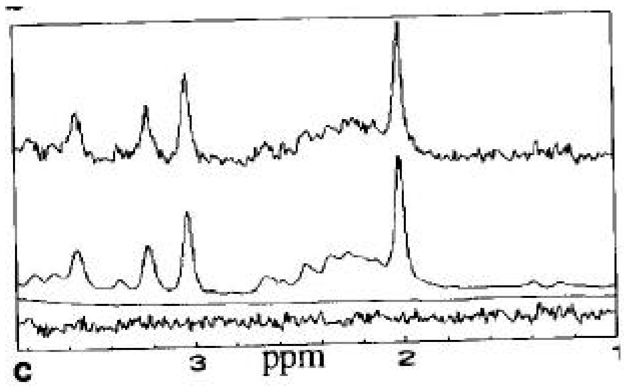
From here onward, we will mostly just consider the magnetization in the rotating frame.

**Magnetic Field Non-Uniformity**

There many things that can affect the magnetic field. These include main

- Magnetic field inhomogeneity – this reflects our inability to make the field perfectly homogeneous. Most magnets are "shimmed" to about 0.5 parts per million over the size of a human head.
- Magnetic susceptibility – this is the magnetization of tissue itself. Different tissues, bones and the surrounding air all have magnetic susceptibility differences of several part per million. The net field is given as $B = B_0(1+c)$, where $c$ is the magnetic susceptibility ($c_{\text{air}}$ is nearly 0, $c_{\text{water}}$ is about $-9\text{x}10^{-6}$ or –9 ppm).
- Chemical shift – Different chemical species have differing shielding of the nucleus from the surrounding electron clouds. Here the net field is $B = B_0(1-s)$, where $s$ is the chemical shift (a positive chemical shift implies shielding of the nucleus or a downward shift in the field). A common chemical shift the shift between water protons (bonded to O) and fat protons (bonded to C): $s_{\text{wf}}$ is about 3.35 ppm. At 1.5 T, this results in a shift of the resonant frequency of about 215 Hz. Below is a proton spectrum in the human head with chemical shift along the x-axis with the biggest 3 peaks being N-acetetyl apartate (NAA), Creatine, and Choline (water has been suppressed and there is no fat in

the middle of the head):



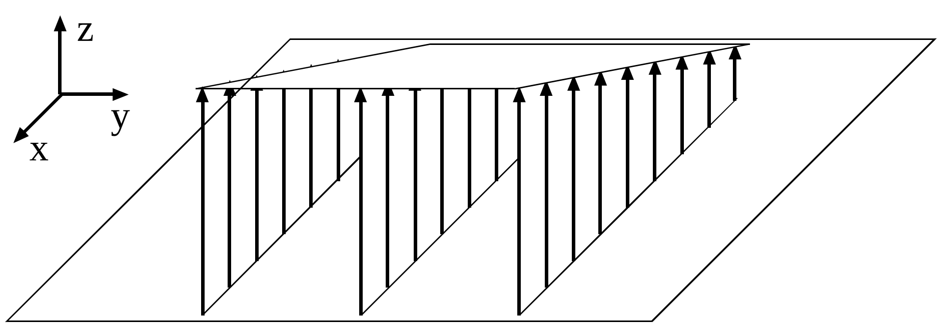

- Gradients – These are intentional linear variations in the magnetic field.

## Gradients

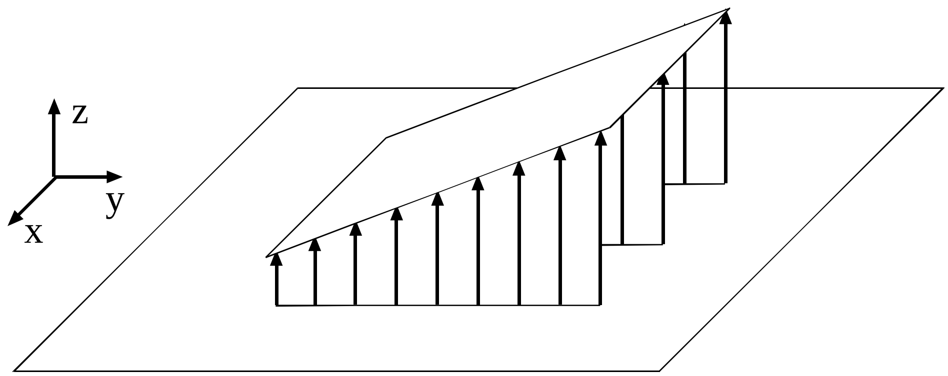
*Gradient fields are the principle tool for localization in MRI*.

It is important to remember the gradient fields vary along some spatial direction, but that field lines are aligned to the main magnetic field. For example:

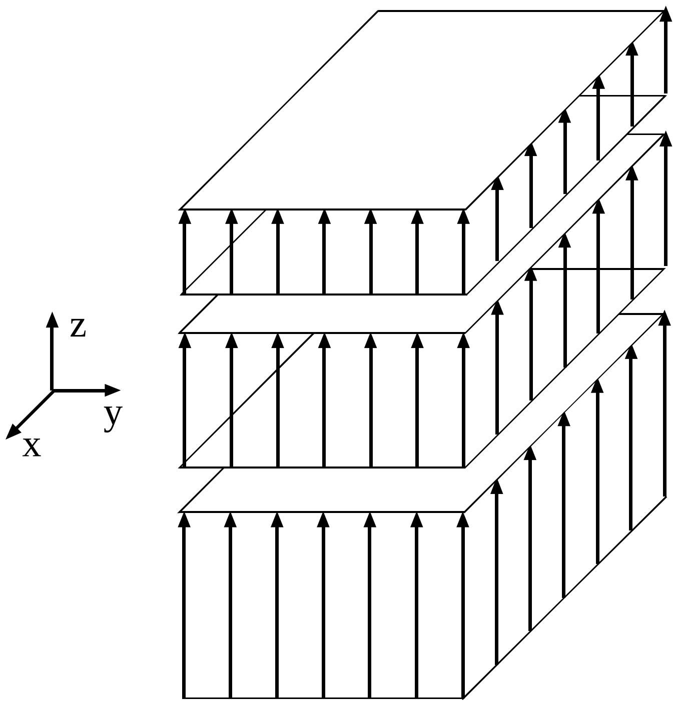
The X-Gradient



The Y-Gradient




The Z-Gradient




**1D Localization**

Let's look at the example of a constant, linear variation in the applied field (known as a "gradient"). Specifically, let the variation be the *x* direction, $\Delta B(x, y, t) = G_x \cdot x$, then the solution to the Bloch equation is:

$$m_{xy,rot}(x,y,t) = m_0(x,y)\exp(-igG_x xt)$$

$$= m_0(x,y)\exp(-i\Delta w(x)t)$$

where the spins will precess at a frequency related to *x* location,

$$\Delta w(x) = gG_x x$$

or

$$\Delta f(x) = \frac{g}{2p} G_x x$$

*Important!* Notice in the preceding expression that frequency has a one-to-one correspondence to spatial location in *x*.

The signal equation for this example is:

$$s(t) = \iint m_0(x,y)\exp\left(-i\Delta w(x)t\right)dxdy$$

$$= \iint m_0(x,y)\exp(-igG_x xt)dxdy$$

Now, let's define

$$m(x) = \int m_0(x,y)dy$$

a function that represents the integral (over y) of all magnetization at each x location. Here the signal is:

$$s(t) = \int m(x)\exp(-igG_x xt)dx$$

Now, if we substitute $gG_x t = 2ps$, we can see that s(t) is really just the 1D FT of $m(x)$:

$$s(t) = \int m(x)\exp(-i2psx)dx\Big|$$

$$= F\{m(x)\}\Big|_{s=gG_x t/2p}$$

$$= M(s = gG_x t/2p)$$

Now, if we want to determine m(x) (recall, our goal in MRI is to make images of the magnetization), then it seems logical to take the inverse FT of the received signal. Rewriting the above relationship we get:

$$M(s) = s(t)\Big|_{t=2ps/gG_x}$$

and now:

$$\hat{m}(x) = F^{-1}\{M(s)\} = F^{-1}\left\{ s\left(\frac{2ps}{gG_x}\right)\right\}$$

Recalling, several of our FT relationship, we can also show that:

$$\hat{m}(x) = \frac{gG_x}{2p} F\{s(t)\}_{f=-gG_x x/2p}$$

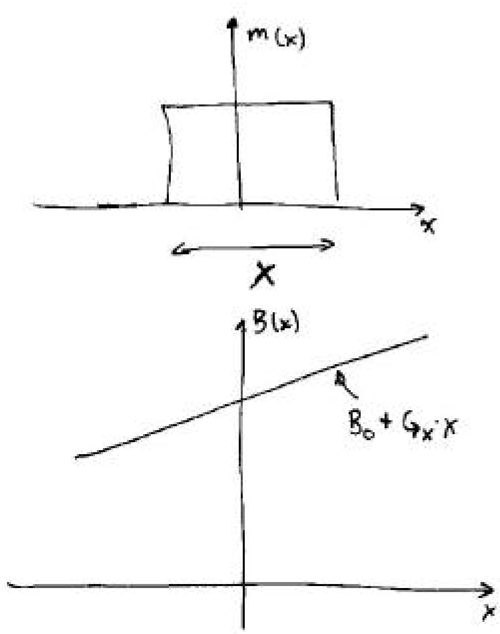
This is the same relationship between frequency and spatial position described before. The negative sign comes from the fact the spins precess in the negative direction (e.g. the negative sign in exp(-i*wt*)).

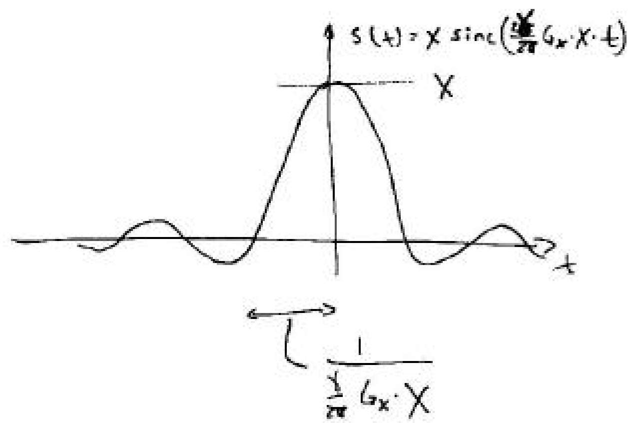
Let's look at an example with *m*(*x*) = rect(*x*/*X*). The received signal will now be:

$$s(t) = F\{m(x)\}\big|_{s=gG_x t/2p}$$

$$= X\text{sinc}(Xs)\ \big|_{s=gG_x t/2p}$$

$$= X\text{sinc}\left( X \frac{g}{2p} G_x t \right)$$



The 1D FT of $s(t)$ will be:

$$\hat{m}(x) = \frac{gG_x}{2p} F\{s(t)\}_{f=-gG_x x/2p}$$

$$= \frac{gG_x}{2p} F\left\{ X\mathrm{sinc}\left( X \frac{g}{2p} G_x t \right) \right\}_{f=-gG_x x/2p}$$

$$= \frac{gG_x}{2p} X \frac{2p}{gG_x X} \mathrm{rect}\left( \frac{2p}{gG_x X} f \right)_{f=-gG_x x/2p}$$

$$= \mathrm{rect}\left( \frac{2p}{gG_x X} \cdot \frac{-gG_x x}{2p} \right) = \mathrm{rect}\left( \frac{x}{X} \right)$$

(same as the original object).