Signal and Image Processing for MRI

Fourier Transformation

- The Fourier transform (FT) of a time-domain function g(t) is a frequency-domain function G(ν), or spectrum

$$G(\nu) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi\nu t} dt$$

•Extract from g(t) the amplitude of the frequency component at frequency ν.
•The inverse FT (IFT) describes the synthesis of a time domain signal from sinusoidal components:

$$g(t) = \int_{-\infty}^{\infty} G(\nu) e^{j2\pi\nu t} d\nu$$

time, t, and frequency, ν, form a FT pair

Fourier Transformation

- In MRI the FT pair: spatial position vectors, **x** = (x, y), and spatial frequency vectors, **k** = (k_x, k_y)
- What is the unit of spatial frequency?

$$G(k_x, k_y) = \iint g(x, y) e^{-j2\pi(k_x x + k_y y)} dx dy$$

$$g(x, y) = \iint G(k_x, k_y) e^{j2\pi(k_x x + k_y y)} dk_x dk_y$$

Sufficient conditions for *g(x,y)*
•Continuous
•integrable

Properties of FT

- Linear: $\mathrm{FT}[a_1 g_1(x, y) + a_2 g_2(x, y)] = a_1 \mathrm{FT}[g_1(x, y)] + a_2 \mathrm{FT}[g_2(x, y)]$
- Translation $\mathrm{FT}[g(x - x_0, y - y_0)] = \mathrm{G}[u, v] e^{-i(ux_0 + vy_0)}$
- Scale $\mathrm{FT}[g(|a|x, |b|y)] = \frac{1}{|ab|} \mathrm{G}[\frac{u}{|a|}, \frac{v}{|b|}]$

Convolution

$$\mathrm{FT}[f(x,y)] = \mathrm{F}[u,v]$$
$$\mathrm{FT}[g(x,y)] = \mathrm{G}[u,v]$$
then
$$\mathrm{FT}[\iint f(x,y)g(x'-x,y'-y)dxdy] = F(u,v)G(u,v)$$

HW: Prove it

Dirac delta function

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)dx = 1 \qquad \textit{unit} \qquad \textit{response}$$

2*D*

$$\delta(x,y) = \delta(x)\delta(y)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)dxdy = 1$$

Sifting

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x-x_0)\delta(y-y_0)dxdy = f(x_0,y_0)$$

FT(δ) = δ

k-Space

$$s(t) = \iint m_{xy,rot}(x,y,t)dxdy$$

Consider a spatially and temporally varying applied magnetic fields introduced by time varying gradient fields:

$$B(x,y,t) = B_0 + G_x(t)\cdot x + G_y(t)\cdot y$$

Then for a rotating frame

$$\Delta\omega(x,y,t) = \gamma\left(G_x(t)\cdot x + G_y(t)\cdot y\right)$$

and

$$\phi(x,y,t) = \int_0^t \gamma\left(G_x(\tau)\cdot x + G_y(\tau)\cdot y\right)d\tau$$

K Space (cont'd)
Signal equation

$$s(t) = \iint m_{xy,rot}(x,y,t)dxdy$$
$$= \iint m(x,y)\exp(-i\phi(x,y,t))dxdy$$
$$= \iint m(x,y)\exp\left(-i\int_0^t \gamma\left(G_x(\tau)\cdot x + G_y(\tau)\cdot y\right)d\tau\right)dxdy$$
$$= \iint m(x,y)\exp\left(-i\gamma\left(\int_0^t G_x(\tau)d\tau\cdot x + \int_0^t G_y(\tau)d\tau\cdot y\right)\right)dxdy$$

with

$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$
$$k_y(t) = \frac{\gamma}{2\pi}\int_0^t G_y(\tau)d\tau$$

$$s(t) = \iint m(x,y)\exp\left(-i2\pi\left(xk_x(t) + yk_y(t)\right)\right)dxdy$$
$$= F_{2D}\{m(x,y)\}|_{u=k_x(t),v=k_y(t)} = M\left(k_x(t),k_y(t)\right)$$

$$s(t) = \mathrm{FT}\{m(x,y)\} = M\left(k_x(t),k_y(t)\right)$$

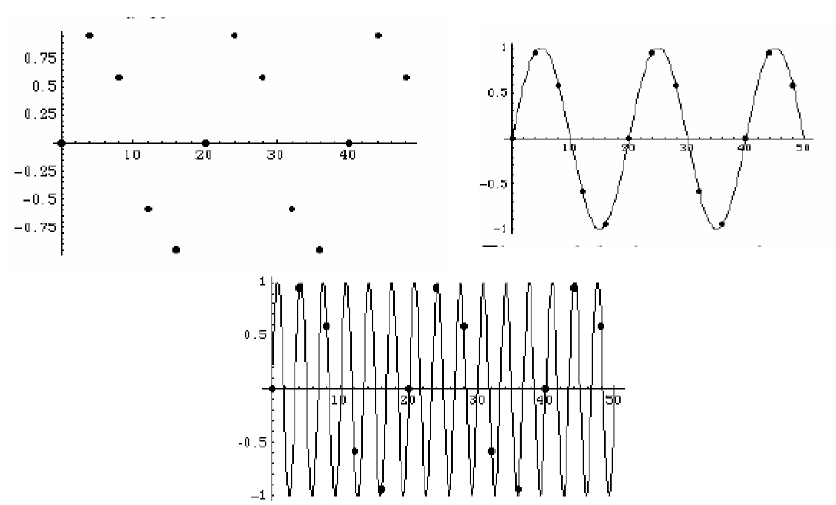
→ the signal is equal to the Fourier transform of the initial magnetization evaluated at locations defined by the *k-space*

- So
  - What is k space
  - Where does it start
  - What control the k-space trajectory
  - Ok we sample FT of the magnetization, how do we get the image

Sampling in *k* space and Aliasing

- Aliasing: An aliased frequency is a high frequency temporal or spatial signal component that is represented at an low frequency. This results from sampling at too low a rate to faithfully capture high frequency components.

Aliasing



Nyquist criterion

- For a bandlimited time-domain signal with highest frequency component v_{max}, aliasing will not occur if the sampling rate, v_s, satisfies $\nu_s > 2\nu_{max}$

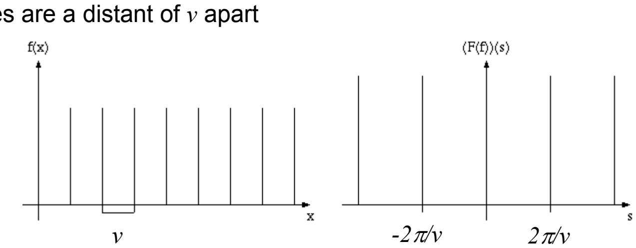
DFT

$$G(\tfrac{n}{N\Delta T}) = \sum_{m=0}^{N-1} g\left(m\Delta T\right) e^{-j2\pi nm/N}$$

Digital convolution of two data streams of length N, x1[n] and x2[n], is defined by

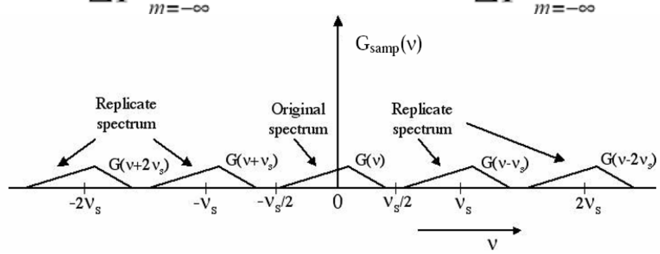
$$c[n] = x_1[n] * x_2[n] \equiv \sum_{k=0}^{N-1} x_1[k]\, x_2[n-k]$$

Comb Functions: infinite series of equidistant Dirac impulses, where adjacent impulses are a distant of v apart



$$g_{samp}(t) = g(t) \bullet \sum_{n=-\infty}^{+\infty} \delta(t - n\Delta T) = \frac{1}{\Delta T} \sum_{m=-\infty}^{m=+\infty} g(t) e^{j2\pi m v \Delta T}$$

$$G_{samp}(v) = \frac{1}{\Delta T} \sum_{m=-\infty}^{m=+\infty} G(v - m/\Delta T) = \frac{1}{\Delta T} \sum_{m=-\infty}^{m=+\infty} G(v - m v_s)$$



The original spectrum is preserved without distortion as long as the frequency spectrum G(v) is bandlimited to frequencies |v| < vs/2 and this is the Nyquist criterion

Sampling in MRI

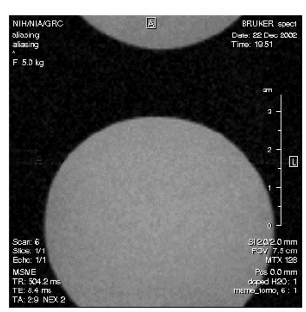
- Sampling of *k* space must be of high enough frequency to properly represent high-frequency spatial components.
- Consider the read direction

$\gamma G_r FOV_r$ = Sampling Bandwidth = $1/\Delta T$

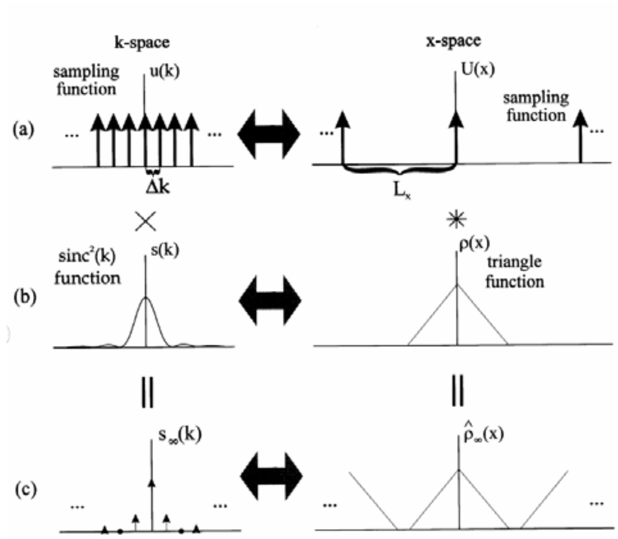
but $\Delta k_r = \gamma G_r \Delta T$

$\Delta T < 1/(\gamma G_r W_r)$

$FOV_r = 1/\Delta k_r$



Sampling concept



Sampling MR *k* space

$$\widetilde{M}(u,v) = M(u,v)\mathrm{comb}\left(\frac{u}{\Delta k_x}, \frac{v}{\Delta k_y}\right)$$

$$= \Delta k_x \Delta k_y \sum_{n,m=-\infty}^{\infty} \delta(u - n\Delta k_x, v - m\Delta k_y) M(n\Delta k_x, m\Delta k_y)$$

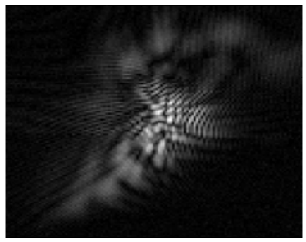
The image (space) domain equivalent is:

$$\widetilde{m}(x,y) = m(x,y) ** \Delta k_x \Delta k_y \mathrm{comb}(\Delta k_x u, \Delta k_y v)$$

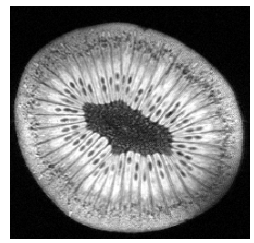
$$= m(u,v) ** \sum_{n,m=-\infty}^{\infty} \delta\left(u - \frac{n}{\Delta k_x}, v - \frac{m}{\Delta k_y}\right)$$

$$= \sum_{n,m=-\infty}^{\infty} m\left(u - \frac{n}{\Delta k_x}, v - \frac{m}{\Delta k_y}\right)$$

$FOV_x = 1/\Delta k_x$ and $FOV_y = 1/\Delta k_y$



IFFT2



k space

image space

K space